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Indivisibility of balls in Euclidean *n*-space

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Abstract

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An open or closed ball in Euclidean n-space cannot be partitioned into k pairwise congruent sets if $2 \leqslant k \leqslant n$.

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1. Introduction

Two subsets of Euclidean n-space $\mathbb{R}^n$ are called *congruent* if there is an isometry of $\mathbb{R}^n$ that maps one onto the other. A subset of $\mathbb{R}^n$ is called *k-divisible* if it can be partitioned into k pairwise congruent sets. $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

Ruziewicz has shown $\mathbb{R}^1$ is k-divisible for $2 \leqslant k \leqslant \mathfrak{c}$, [5]. It is obvious that each half-open interval is k-divisible for $2 \leqslant k < \aleph_0$. Von Neumann has shown that each interval, whether open, closed or half-open, is $\aleph_0$-divisible, [4]. This was generalized by Mycielski who showed that all those intervals are κ-divisible for any κ such that $\aleph_0 \leqslant \kappa \leqslant 2^{\aleph_0}$, [3]. By contrast, Sierpiński has shown that open intervals are not 2-divisible, [6], and Gustin has shown that open and closed intervals are not k-divisible if $2 \leqslant k < \aleph_0$, [2]. Wagon has conjectured that more generally open or closed balls in Euclidean n-space are not 2-divisible, and has proved this for $n = 2, 3$, [7].

In this paper we establish Wagon's conjecture:

Theorem 1.1. *If $\mathscr{A}$ is a cover of a ball in $\mathbb{R}^n$ with congruent sets and only one of them contains the center of this ball, then there are at least $n+1$ sets in $\mathscr{A}$.*

This motivates the following extension of Wagon's conjecture.

Conjecture 1.2. If $2 \leq k < \aleph_0$ then open balls and closed balls in $\mathbb{R}^n$ are not k-divisible.

We are too cautious to make conjectures for $k = \aleph_0$, and just ask this:

Question 1.3. If $n \geq 2$, and open balls and closed balls in $\mathbb{R}^n$ not $\aleph_0$-divisible?

2. Balls

Let $n \geq 2$ and consider any $B \subseteq \mathbb{R}^n$ satisfying

$$\{x \in \mathbb{R}^n : \|x\| < 1\} \subseteq B \subseteq \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

Consider any cover $\mathscr{A}$ of B into pairwise congruent sets with $|\mathscr{A}| \geq 2$ such that only one element of $\mathscr{A}$ contains the center of B. We will prove $|\mathscr{A}| \geq n+1$. Let C be the member of $\mathscr{A}$ that contains the center 0 of B. For any $A \in \mathscr{A}$ choose an isometry σ_A of $\mathbb{R}^n$ with $\sigma_A^{\to} C = A$. Clearly

(a) $(\forall A \in \mathscr{A} \backslash \{C\})[\sigma_A(0) \neq 0]$.

The key to our argument is the observation that

(b) $(\forall A \in \mathscr{A})[\operatorname{diam}(A) < 2]$.

For the proof we establish the following inequality, which we have not bothered to try to find in the literature.

(c) $(\forall p, q, x, y \in \mathbb{R}^n)[\|x-y\|^2 + \|p-q\|^2 \leq \|x-p\|^2 + \|p-y\|^2 + \|y-q\|^2 + \|q-x\|^2]$.

Since $(x+y, p+q) \leq \|x+y\| \, \|p+q\|$ and $0 \leq \|x+y\| - (\|p+q\|)^2$ we have

$$\begin{aligned} &2((x,p)+(p,y)+(y,q)+(q,x)) \\ &\quad \leq \|x\|^2 + \|y\|^2 + 2(x,y) + \|p\|^2 + \|q\|^2 + 2(p,q), \end{aligned}$$

hence

$$\begin{aligned} &\|x\|^2 + \|y\|^2 - 2(x,y) + \|p\|^2 + \|q\|^2 - 2(p,q) \\ &\quad \leq 2(\|x\|^2 + \|y\|^2 + \|p\|^2 + \|q\|^2 - (x,p) - (p,y) - (y,q) - (q,x)), \end{aligned}$$

from which (c) follows.

Since each σ_A is an isometry we prove (b) if we pick any $A \in \mathscr{A} \backslash \{C\}$ and prove $\operatorname{diam}(A) < 2$: Indeed, pick $x, y \in A$, then $\|x\|, \|y\| \leq 1$ since $x, y \in B$, and also $\|x - \sigma_A(0)\|, \|y - \sigma_A(0)\| \leq 1$ since $x, y \in \sigma_A^{\to} C \subseteq \sigma_A^{\to} B$. So applying (c) with $p = 0$, $q = \sigma_A(0)$ we find $\|x-y\|^2 + \|\sigma_A(0)\|^2 \leq 4$. This proves (b) since $\|\sigma_A(0)\| \neq 0$ because of (a).

Now (b) immediately implies $|\mathscr{A}| \geqslant n+1$ because of the Lusternik–Schnirelman Theorem, which says that if $\mathscr{F}$ is a collection of closed sets such that $|\mathscr{F}| \leqslant k+1$ and $\bigcup \mathscr{F} = S^k = \{x \in \mathbb{R}^{k+1} \colon \|x\| = 1\}$ then $(\exists F \in \mathscr{F})[\mathrm{diam}(F) = 2]$, [1, XVI, 6.2(3)]: just assume $|\mathscr{A}| \leqslant n$, let $k = n-1$ and $\mathscr{F} = S^{n-1} \cap \bar{A} \colon A \in \mathscr{A}\}$ ($\bar{A}$ = closure of A) to get a contradiction with (b).

We think it is of interest to have an elementary proof that $|\mathscr{A}| \geqslant n+1$.

Again we argue by contradiction, and assume $|\mathscr{A}| \leqslant n$. By (a), if $\alpha = \min_{A \in \mathscr{A}\setminus\{C\}} \|\sigma_A(0)\|$ then $\alpha > 0$. For $A \in \mathscr{A}\setminus\{C\}$ the set $\{x \in \mathbb{R}^n \colon (x, \sigma_A(0)) = 0\}$ is an $(n-1)$-dimensional hyperplane. Since $|\mathscr{A}\setminus\{C\}| \leqslant n-1$ and $\alpha > 0$ it follows

$$H = \{x \in \mathbb{R}^n \colon (\forall A \in \mathscr{A}\setminus\{C\})[(x, \sigma_A(0)) = 0]\}$$

is a subspace of dimension at least 1. Since $\alpha > 0$ it follows

$$H' = \{x \in H \colon 1 - \alpha^2 < \|x\| < 1\}$$

has $\mathrm{diam}(H') = 2$ (since $(\forall x \in H')[-x \in H']$), hence $H' \not\subseteq C$ because of (b). Now for each $x \in H'$ and $A \in \mathscr{A}\setminus\{C\}$, since $(x, \sigma_A(0)) = 0$ we have $\|x - \sigma_A(0)\|^2 = \|x\|^2 + \|\sigma_A(0)\|^2 > 1 - \alpha^2 + \alpha^1 = 1$, hence $x \notin A$ since $A = \sigma_A^{\to} C \subseteq \sigma_A^{\to} B$. As $H' \subseteq B = \bigcup \mathscr{A}$ it follows $H' \subseteq C$, an absurdity.

This completes the proof of Theorem 1.1.

Virtually the same proof shows that B is not k-divisible for any finite $k \geqslant 2$ if instead of $\mathbb{R}^n$ we take any infinite-dimensional pre-Hilbert space.

Unfortunately, if B is open or closed there is a cover of B by $n+1$ pairwise congruent sets only one of which contains 0.

Remark by the referee. The manuscript on which this paper is based also contains a proof of the result due to Gustin [2] that open intervals and closed intervals are not k-divisible if $2 \leqslant k < \aleph_0$. Since van Douwen's proof of this result is not significantly shorter than Gustin's proof, it was decided by the editor not to publish it and to update the introduction of the present paper.

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